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On linear preservers of (right) matrix majorization

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Abstract

An $n \times m$ matrix A is said to be *matrix majorized* (or more precisely matrix majorized from the right) by an $n \times m$ matrix B, and write $A \prec B$, if there exists a row stochastic matrix R such that $A = BR$. We characterize the linear operators that preserve the matrix majorization $\prec$.

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1. Introduction

A nonnegative real matrix R with the property that all its row sums are +1 is said to be *row stochastic*. The following notation will be fixed throughout the paper: M_{nm} for the collection of all $n \times m$ real matrices, $M_n = M_{nn}$ for the collection of all $n \times n$ real matrices, $\mathscr{RS}(n)$ for the set of all $n \times n$ row stochastic matrices, $\mathscr{P}(n)$ for the set of all $n \times n$ permutation matrices, $\mathbb{R}^n$ for the set of all real $n \times 1$ (column) vectors, and $\mathbb{R}_m$ for the set of all real $1 \times m$ (row) vectors.

Let $A, B \in M_{nm}$. Then A is said to be *matrix majorized* by B, and is denoted as $A \prec B$, if $A = BR$ for some row stochastic matrix R.

The notation $[X_1/X_2/\cdots/X_n]$ is used for the $n \times m$ matrix whose rows are $X_1, X_2, \ldots, X_n \in \mathbb{R}_m$. If $X_1 = X_2 = \cdots = X_n = a \in \mathbb{R}_m$ then $[X_1/X_2/\cdots/X_n]$ is denoted by $a^{(n)}$. The standard

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basis of $\mathbb{R}_m$ will be denoted by $\{e_1, e_2, \ldots, e_m\}$ and that of $\mathbb{R}^n$ by $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. The transpose of a matrix $X \in M_{nm}$ is denoted by X^t, which belongs to M_{mn}.

Let $\mathscr{A}$ be a linear space of matrices, let T be a linear operator on $\mathscr{A}$, and let $\mathscr{R}$ be a relation on $\mathscr{A}$. We say T *preserves* $\mathscr{R}$ if

$$\mathscr{R}(T(X), T(Y)) \quad \text{whenever } \mathscr{R}(X, Y).$$

In the next section, we characterize the linear operators that preserve the matrix majorization $\prec$.

For more information on the type of majorization discussed in the present paper as well as other types of majorization we refer the reader to [1–7]. The following characterization of the simpler matrix majorization $A \prec_\ell B$ defined by $A = RB$ for some $R \in \mathscr{RS}(n)$ (depending on A, B) is known and will be used in the proof of our main results.

Theroem 1.1 ([4]). *Let $T : M_{nm} \longrightarrow M_{nm}$ be a linear operator. Then T preserves $\prec_\ell$ if and only if $T(X) = (aI + bQ)XL$ for all $X \in M_{nm}$, where $L \in M_m$, $Q \in \mathscr{P}(n)$, $Q \neq I$, and a and b are real numbers such that $ab \leqslant 0$, and, if $n \neq 2$, $ab = 0$. In case $n \neq 2$, $aI + bQ = cP$ for some $c \in \mathbb{R}$ and some $P \in \mathscr{P}(n)$ and, hence, $T(X) \equiv PXK$ for some $K \in M_m$.*

2. Main results

In this section we prove the main result of the paper. In what follows, for a matrix $X = [x_{ij}] \in M_{nm}$, its average (column) vector $\overline{X} = [\overline{X}_1/\overline{X}_2/\cdots/\overline{X}_n] \in \mathbb{R}^n$ is defined by the components $\overline{X}_i = m^{-1}(x_{i1} + x_{i2} + \cdots + x_{im})$ $(i = 1, 2, \ldots, n)$. In particular, if $a = [a_1, a_2, \ldots, a_m] \in \mathbb{R}_m$, we define $\bar{a} = m^{-1}(a_1 + a_2 + \cdots + a_m)$. First, we study the case $n = 1$.

Theroem 2.1. *If a linear operator $T : \mathbb{R}_m \longrightarrow \mathbb{R}_m$ preserves $\prec$, then one of the following conditions* (i), (ii) *or* (iii) *holds*:

(i) rank$(T) \leqslant 1$ *and, in fact, there exists $a \in \mathbb{R}_m$ such that*

$$T(X) = Xa^{(m)} = ma\overline{X} \quad \textit{for all } X \in \mathbb{R}_m. \tag{1}$$

(ii) rank$(T) = m = 2$ *and*

$$T(X) = \ell X(cI + dQ) \quad \textit{for all } X \in \mathbb{R}_m, \tag{2}$$

where $0 \neq \ell \in \mathbb{R}$, $Q = [[0, 1]/[1, 0]]$, *and* $0 \leqslant \min\{c, d\} < \max\{c, d\} = 1$.

(iii) rank$(T) = m \geqslant 3$ *and*

$$T(X) = \ell XP \quad \textit{for all } X \in \mathbb{R}_m, \tag{3}$$

where $0 \neq \ell \in \mathbb{R}$ *and* $P \in \mathscr{P}(m)$.

Moreover, any $A \in M_m$ satisfying $T(X) \equiv XA$ is the matrix $a^{(m)}$ of (i), *$\ell(cI + dQ)$ of* (ii), *or ℓP of* (iii), *accordingly.*

Note. Cases (ii) and (iii) can be unified by writing $P = cI + dQ$, for appropriate choices of $c, d \in \{0, 1\}$ and $I \neq Q \in \mathscr{P}(m)$.

Proof. The case $m = 1$ being trivial, we let $m \geqslant 2$. Assume the linear operator $T : \mathbb{R}_m \longrightarrow \mathbb{R}_m$ preserves $\prec$, and let $A = [A_1/A_2/\cdots/A_m]$ be its matrix representation with respect to the standard basis $\{e_1, e_2, \ldots, e_m\}$ of $\mathbb{R}_m$. This means that $T(X) = XA$ for all $X \in \mathbb{R}_m$. Consider the following two cases.

Case (1). *Assume A is not invertible*: Choose a nonzero $C = [c_1, c_2, \ldots, c_m] \in \mathbb{R}_m$ such that $c_1A_1 + c_2A_2 + \cdots + c_mA_m = 0$. Then $T(C) = CA = 0$. Define $C' = [c_+, -c_-, 0, \ldots, 0] \in \mathbb{R}_m$ by $c_\pm = \sum c_i^\pm$, where $u^\pm = (|u| \pm u)/2$ for $u \in \mathbb{R}$. Choose $R = [R_1/R_2/\cdots/R_m] \in \mathscr{RS}(m)$ such that $R_i = e_1$ if $c_i \geqslant 0$ and $R_i = e_2$ if $c_i < 0$ $(i = 1, \ldots, m)$. Then $C' = CR$ and, hence, $C'P = CRP$ for all $P \in \mathscr{P}(m)$. Also, for all $P \in \mathscr{P}(m)$, $RP \in \mathscr{RS}(m)$ and $C'PA = T(C'P) = T(CRP) = T(C)S = CAS = 0$ for some $S \in \mathscr{RS}(m)$. In particular, $c_+A_r - c_-A_s = 0$ for every pair $r \neq s$. It follows that, $A_1 = A_2 = \cdots = A_m = a$ for some $a \in \mathbb{R}_m$. Then $A = a^{(m)}$ and T is of the form (i).

Case (2). *Assume A is invertible*: Define $\widetilde{T} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by $\widetilde{T}(X) = A^{-1}X$ for all $X \in \mathbb{R}^m$. Let $R \in \mathscr{RS}(m)$ be arbitrary. Then $\widetilde{T}(RX) = A^{-1}RX = A^{-1}RAA^{-1}X = (A^{-1}RA)\widetilde{T}(X)$ for all $X \in \mathbb{R}^m$. We claim $A^{-1}RA \in \mathscr{RS}(m)$. To prove this, we show that, for each i, the ith row of $A^{-1}RA$ is a row of some row stochastic matrix. In fact, $e_iA^{-1}RA = T(e_iA^{-1}R) = T(e_iA^{-1})S = e_iS$ for some $S \in \mathscr{RS}(m)$ (depending on i). Thus, $\widetilde{T} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ preserves $\prec_\ell$, where the relation $X \prec_\ell Y$ $(X, Y \in \mathbb{R}^m)$ means that $X = RY$ for some $R \in \mathscr{RS}(m)$ depending on X, Y. It now follows from Theorem 1.1 that either $m = 2$ and $\widetilde{T}(X) \equiv k(aI + bQ)X$, or $m \geqslant 3$ and $\widetilde{T}(X) \equiv kP_1X$, where $0 \neq k \in \mathbb{R}$, $a, b \in \mathbb{R}$, $ab \leqslant 0$, $Q = [[0, 1]/[1, 0]]$, and $P_1 \in \mathscr{P}(m)$.

In case $m = 2$, $A^{-1} = k(aI + bQ)$ and, hence, $A = \ell(cI + dQ)$, where $0 \neq \ell \in \mathbb{R}$, $0 \leqslant \min\{c, d\} < \max\{c, d\} = 1$. In case $m \geqslant 3$, $A^{-1} = kP_1$ and, hence, $A = k'P$, where $P = P_1^{\mathrm{t}}$ (the transpose of P_1). Thus, if A is invertible, then T is of the form (2) or (3).

The last part of the theorem follows from the fact that the matrix representation of T with respect to the standard basis of $\mathbb{R}_m$ is unique. □

We now study the general case of our main result.

Theroem 2.2. *A linear operator $T : M_{nm} \longrightarrow M_{nm}$ preserves the matrix majorization $\prec$, if and only if T satisfies one of the following conditions* (i), (ii), *or* (iii).

(i) *There exists $\mathscr{A} \in M_n(\mathbb{R}_m)$ such that*

$$T(X) = m\mathscr{A}\overline{X} \quad \textit{for all } X \in M_{nm};$$

that is,

$$T([X_1/X_2/\cdots/X_n]) = m\left[\sum_{j=1}^{n} a_{1j}\overline{X}_j \Big/ \sum_{j=1}^{n} a_{2j}\overline{X}_j \Big/ \cdots \Big/ \sum_{j=1}^{n} a_{nj}\overline{X}_j\right],$$

where $a_{ij} \in \mathbb{R}_m$ is the (i, j) entry of $\mathscr{A}$.

(ii) *$m = 2$ and there exist $L \in M_n$ and $c, d \in \mathbb{R}$ such that*

$$T(X) = LX(cI + dQ) \quad \textit{for all } X \in M_{nm},$$

and $0 \leqslant \min\{c, d\} < \max\{c, d\} = 1$.

(iii) *$m \geqslant 3$ and there exist $L \in M_n$ and $P \in \mathscr{P}(m)$ such that*

$$T(X) = LXP \quad \textit{for all } X \in M_{nm}.$$

Proof. We first prove the sufficiency of the condition. Let T be as in (i) and let $R \in \mathscr{RS}(m)$ be arbitrary. It is easy to check that $\overline{XR} = \overline{X}$ for all $X \in M_{nm}$. Thus $T(XR) = m\mathscr{A}(\overline{XR}) = m\mathscr{A}\overline{X} = T(X)$ and hence T preserves $\prec$. Next, assume T satisfies (ii). Then $T(XR) = LXR(cI + dQ) = LX(cI + dQ)S = (T(X))S$ for all $R \in \mathscr{RS}(2)$, where $S = (cI + dQ)^{-1}R(cI + dQ) \in \mathscr{RS}(2)$. (Straightforward calculation establishes the last claim for S.) Finally, if T satisfies (iii), then $T(XR) = LXRP = LXPS$ for all $R \in \mathscr{RS}(m)$, where $S = P^{-1}RP \in \mathscr{RS}(m)$. This completes the proof of the sufficiency.

We now prove the necessity of the condition. For each $i = 1, 2, \ldots, n$, define $E_i : \mathbb{R}_m \longrightarrow M_{nm}$ by $E_i(x) = \varphi_i x$ for all $x \in \mathbb{R}_m$. Also, define $E^i : M_{nm} \longrightarrow \mathbb{R}_m$ by $E^i(X) = \varphi_i^t X$ for all $X \in M_{nm}$. (Recall that $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is the standard basis of $\mathbb{R}^n$.) Then each linear operator $T_{ij} = E^j T E_i : \mathbb{R}_m \longrightarrow \mathbb{R}_m$ preserves $\prec$ and, by Theorem 2.1, is of the form (1), (2) or (3). We distinguish the following two cases.

Case (1). *Assume every T_{ij} is noninvertible*: Then $T_{ij}(x) \equiv m a_{ij}\overline{x}$ for all $i, j = 1, 2, \ldots, n$. Then

$$\begin{aligned} T([X_1/X_2/\cdots/X_n]) &= \left[\sum_{j=1}^{n} T_{1j}(X_j) \Big/ \sum_{j=1}^{n} T_{2j}(X_j) \Big/ \cdots \Big/ \sum_{j=1}^{n} T_{nj}(X_j)\right] \\ &= m\left[\sum_{j=1}^{n} a_{1j}\overline{X}_j \Big/ \sum_{j=1}^{n} a_{2j}\overline{X}_j \Big/ \cdots \Big/ \sum_{j=1}^{n} a_{nj}\overline{X}_j\right] \\ &= m\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_n \end{bmatrix}. \end{aligned}$$

Hence T is of the form (i).

Case (2). *Assume $m \geqslant 2$ and there exist some i, j such that T_{ij} is invertible*: Then T_{ij} is of the form (2) or (3). Composing T with appropriate permutation matrices, we can assume without loss of generality that $i = j = 1$. Dividing T by a suitable constant, we can further assume that $T_{11}(x) \equiv x(cI + dQ)$, where $0 \leqslant \min\{c, d\} < \max\{c, d\} = 1$ and $I \neq Q \in \mathscr{P}(m)$. Moreover, $\min\{c, d\} = 0$ if $m \geqslant 3$ (see the note to Theorem 2.1). We will show that $T_{ij} = \ell_{ij}T_{11}$ for every $i, j = 1, 2, \ldots, n$ and, for this, it is enough to assume $i, j \in \{1, 2\}$. We proceed the proof in the following steps.

Step I. We claim $T_{12} = \ell_{12}T_{11}$ for some $\ell_{12} \in \mathbb{R}$. Assume, if possible, $T_{12}(x) \equiv xa^{(m)}$ for some nonzero $a \in \mathbb{R}_m$. Choose $y \in \mathbb{R}_m$ such that $m\overline{y} = 1$. Let $x = -a(cI + dQ)^{-1}$. Then, for every $R \in \mathscr{RS}(m)$,

$$T([x/y/0/\cdots/0]R) = (T([x/y/0/\cdots/0]))S \tag{4}$$

for some $S \in \mathscr{RS}(m)$ and, hence,

$$\begin{aligned} -a(cI + dQ)^{-1}R(cI + dQ) + yRa^{(m)} &= [-a(cI + dQ)^{-1}(cI + dQ) + ya^{(m)}]S \\ &= (-a + a)S = 0. \end{aligned}$$

Since $yRa^{(m)} = ya^{(m)} = m\bar{y}a = a$, it follows that $a(cI + dQ)^{-1}(R - I) = 0$ for all $R \in \mathscr{RS}(m)$. Thus, $a(cI + dQ)^{-1} = 0$; a contradiction. Hence, either $T_{12} = 0$ or there exist $k, c_1, d_1 \in \mathbb{R}$ and $Q_1 \in \mathscr{P}(m)$ such that $0 \leqslant \min\{c_1, d_1\} < \max\{c_1, d_1\} = 1$, $k \neq 0$, $Q_1 \neq I$, and $T_{12}(x) = kx(c_1 I + d_1 Q_1)$ for all $x \in \mathbb{R}_m$. Assume $T_{12} \neq 0$. Let $x \in \mathbb{R}_m$ be arbitrary and suppose $y = -k^{-1}x(cI + dQ)(c_1 I + d_1 Q_1)^{-1}$. Then for every $R \in \mathscr{RS}(m)$ there exists $S \in \mathscr{RS}(m)$ such that (4) holds. Then

$$xR(cI + dQ) + kyR(c_1 I + d_1 Q_1) = [x(cI + dQ) + ky(c_1 I + d_1 Q_1)]S,$$

and, hence,

$$xR(cI + dQ) - x(cI + dQ)(c_1 I + d_1 Q_1)^{-1} R(c_1 I + d_1 Q_1) = 0.$$

Since $x \in \mathbb{R}_m$ is arbitrary,

$$R(cI + dQ)(c_1 I + d_1 Q_1)^{-1} = (cI + dQ)(c_1 I + d_1 Q_1)^{-1} R;$$

also, since $R \in \mathscr{RS}(m)$ is arbitrary, $(cI + dQ)(c_1 I + d_1 Q_1)^{-1} = \lambda I$ for some nonzero $\lambda \in \mathbb{R}$. Hence, $c_1 I + d_1 Q_1 = \ell_{12}(cI + dQ)$.

Step II. We assume T_{21} is noninvertible and show that $T_{21} = 0$. To reach a contradiction, assume T_{21} is nonzero and noninvertible. Then $T_{21}(x) \equiv xa^{(m)}$ for some nonzero $a \in \mathbb{R}_m$. Now, let $t \in \mathbb{R}$, let $R \in \mathscr{RS}(m)$, and let $x = (a + mte_1)(cI + dQ)^{-1}$. Find $S \in \mathscr{RS}(m)$, depending on t and R, such that

$$T([x/0/\cdots/0]R) = (T([x/0/\cdots/0]))S. \tag{5}$$

Then

$$(a + mte_1)(cI + dQ)^{-1} R(cI + dQ) = (a + mte_1)S, \tag{6}$$

$$(a + mte_1)(cI + dQ)^{-1} Ra^{(m)} = (a + mte_1)(cI + dQ)^{-1} a^{(m)} S. \tag{7}$$

Let $z = (a + mte_1)(cI + dQ)^{-1}$. Since $Ra^{(m)} = a^{(m)}$ and since $\bar{a} + t = \overline{a + mte_1} = \overline{z(cI + dQ)} = \overline{cz + dzQ} = c\bar{z} + d\bar{z} = (c + d)\bar{z}$, Eq. (7) simplifies as $(\bar{a} + t)a = (\bar{a} + t)aS$. Thus $a = aS$ for all $t \neq -\bar{a}$. Now, substituting this in (6) and letting $t \longrightarrow 0$ yields $a(cI + dQ)^{-1}R(cI + dQ) = a$ or, equivalently, $a(cI + dQ)^{-1}R = a(cI + dQ)^{-1}$ for all $R \in \mathscr{RS}(m)$; a contradiction.

Step III. We assume T_{22} is noninvertible and show that $T_{22} = 0$. Assume T_{22} is nonzero, to reach a contradiction. By Step I, $T_{21} = 0$, and by Step II, $T_{12} = 0$. Let $T_{22}(x) \equiv xb^{(m)}$. Choose $x = (b + mte_1)(cI + dQ)^{-1}$, and $y = b + mte_1$ for $t \neq -\bar{b}$ and insert them in the following equation:

$$T([x/y/0/\cdots/0]R) = (T([x/y/0/\cdots/0]))S, \tag{8}$$

where R runs through $\mathscr{RS}(m)$ and S is accordingly chosen from $\mathscr{RS}(m)$.

Eq. (8) yields the following system of two equations:

$$xR(cI + dQ) = x(cI + dQ)S,$$

and

$$yRb^{(m)} = yb^{(m)}S.$$

Hence, $b = bS$ for all $t \neq -\bar{b}$ and

$$(b + mte_1)(cI + dQ)^{-1}R(cI + dQ) = b + mte_1 S.$$

Letting $t \longrightarrow 0$ yields $b(cI + dQ)^{-1}R = b(cI + dQ)^{-1}$; a contradiction.

Step IV. We show that $T_{21}(x) \equiv \ell_{21}x(cI + dQ)$. Assume without loss of generality that $T_{21} \neq 0$. Then $T_{21}(x) \equiv \ell_{21}x(c'I + d'Q')$ with $0 \leqslant \min\{c', d'\} \leqslant \max\{c', d'\} = 1$ and $I \neq Q' \in \mathscr{P}(m)$. Let $x = [1, 1, \dots, 1](cI + dQ)^{-1} \in \mathbb{R}_m$ and insert it in (5) to obtain

$$[1, 1, \dots, 1](cI + dQ)^{-1}R(cI + dQ) = [1, 1, \dots, 1]S, \tag{9}$$

$$[1, 1, \dots, 1](cI + dQ)^{-1}R(c'I + d'Q') = [1, 1, \dots, 1](cI + dQ)^{-1}(c'I + d'Q')S. \tag{10}$$

If $m \geqslant 3$, then $cI + dQ = P$ and $c'I + d'Q' = P'$ for some $P' \in \mathscr{P}(m)$. In this case, the right-hand sides of (9) and (10) are equal and, hence,

$$[1, 1, \dots, 1]R = [1, 1, \dots, 1]RP'P^{\mathrm{t}}.$$

Assume $P'P^{\mathrm{t}} \neq I$. Then there exist $p \neq q$ such that $P'P^{\mathrm{t}}\varphi_p = \varphi_q$. Choose $R \in \mathscr{RS}(m)$ with pth column equal to $[1/1/\cdots/1]$. Hence, the pth component of the vector of the left-hand side is m while that of the right-hand side is 0; a contradiction. Thus $P' = P$ and, hence, $T_{21}(x) \equiv \ell_{21}xP$.

If $m = 2$, then $Q = Q' = [[0, 1]/[1, 0]]$ and

$$(cI + dQ)^{-1}(c'I + d'Q') = aI + bQ$$

for some $a, b \in \mathbb{R}$ with $a^2 - b^2 \neq 0$. Also, choose a rank-one matrix $R \in \mathscr{RS}(m)$ such that $(cI + dQ)^{-1}R(cI + dQ) = [[r, 1 - r]/[r, 1 - r]]$ for some $r \neq 1/2$. Now, the fact that the right-hand sides of (9) and (10) having ratio $1/(a + b)$ yields $(a + b)[1, 1][[r, 1 - r]/[r, 1 - r]] = [1, 1][[r, 1 - r]/[r, 1 - r]](aI + bQ)$ and, hence,

$$(a + b)[2r, 2 - 2r] = [2r(a - b) + 2b, 2r(b - a) + 2a].$$

Hence $b = 0$ and, thus, $(c'I + d'Q') = a(cI + dQ)$. This completes the proof of Step IV.

Step V. We show that $T_{22}(x) \equiv \ell_{22}x(cI + dQ)$. Note that if $T_{21} \neq 0$ (resp. $T_{12} \neq 0$), it follows from Step I (resp. Step IV) that $T_{22}(x) \equiv \ell_{22}x(cI + dQ)$. So, we assume $T_{12} = T_{21} = 0$. Let $T_{22}(x) \equiv \ell_{22}x(c'I + d'Q')$ with $0 \leqslant \min\{c', d'\} \leqslant \max\{c', d'\} = 1$ and $I \neq Q' \in \mathscr{P}(m)$. Let $x = y = [1, 1, \dots, 1](cI + dQ)^{-1} \in \mathbb{R}_m$ and insert them in (8) to obtain the system of the two Eqs. (9) and (10) of Step IV. Following the same argument given there, yields a similar result for T_{22}.

Summing up, we have shown in Case 2 that

$$\begin{aligned} T(X) &= T([X_1/X_2/\cdots/X_n]) \\ &= \left[\sum_{j=1}^{n} \ell_{1j}X_j(cI + dQ) \Big/ \cdots \Big/ \sum_{j=1}^{n} \ell_{nj}X_j(cI + dQ)\right] = LX(cI + dQ), \end{aligned}$$

where $L = [\ell_{ij}] \in M_n$, $cd \geqslant 0$, $I \neq Q \in \mathscr{P}(m)$, $c \neq d$ and, if $m \geqslant 3$, $cd = 0$. $\square$

As an immediate consequence of Theorem 2.2, we can state the following corollary. This is a generalization of Theorem 4.4 of [3]. In the following, by a linear strong preserver of matrix majorization, we mean a linear operator $T : M_{nm} \longrightarrow M_{nm}$ such that $TX \prec TY$ if and only if $X \prec Y$ for $X, Y \in M_{nm}$.

Corollary 2.1. *A linear operator $T : M_{nm} \longrightarrow M_{nm}$ strongly preserves the matrix majorization $\prec$ if and only if $T(X) \equiv LXP$ $(X \in M_{nm})$ for a unique $P \in \mathscr{P}(m)$ and a unique invertible L in M_n.*

Proof. It is well known that a linear strong preserver of $\prec$ is bijective. Thus leaving aside the simple case of $m = 1$, we conclude from parts (ii) and (iii) of the theorem that $T(X) \equiv LX(cI + dQ)$ for all $X \in M_{nm}$, where L is further assumed to be invertible. If $m \geqslant 3$, then $cI + dQ = P \in \mathscr{P}(m)$ and we are done. So assume $m = 2$ and let $S = T^{-1}$ and $K = [k_{ij}] = L^{-1}$. Then the components S_{ij} satisfying $S([X_1/X_2/\cdots/X_n]) = \left[\sum_{j=1}^{n} S_{1j}(X_j)/\sum_{j=1}^{n} S_{2j}(X_j)\big/\cdots\big/\sum_{j=1}^{n} S_{nj}(X_j)\right]$ have the form $S_{ij}x \equiv k_{ij}x(cI + dQ)^{-1}$. Choose i, j such that $S_{ij} \neq 0$. Since S is a linear preserver of $\prec$ so is S_{ij} and hence $S_{ij}x \equiv x(c'I + d'Q)$ with $c'd' \geqslant 0$. Then $x(c'I + d'Q) \equiv k_{ij}x(cI + dQ)^{-1}$ and, hence, $(cI + dQ)^{-1} = aI + bQ$ with $ab \geqslant 0$. However, this is possible only when $cd = 0$ and thus $cI + dQ$ is a 2×2 permutation matrix. $\square$

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